Lotka-Volterra with randomly fluctuating environments: a full description

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Abstract

In this note, we study the long time behavior of Lotka-Volterra systems whose coefficients vary randomly. Benaïm and Lobry established that randomly switching between two environments that are both favorable to the same species may lead to four different regimes: almost sure extinction of one of the two species, random extinction of one species or the other and persistence of both species. Our purpose here is to provide a complete description of the model. In particular, we show that any couple of environments may lead to the four different behaviours of the stochastic process depending on the jump rates.

1 Introduction

For a given set of positive parameters $\varepsilon = (a, b, c, d, \alpha, \beta)$, consider the Lotka-Volterra differential system in $\mathbb{R}_+^2$, is given by

$$\begin{cases} x' = \alpha x(1 - ax - by) \\ y' = \beta y(1 - cx - dy) \\ (x_0, y_0) \in \mathbb{R}_+^2 \end{cases}$$

We denote by F_ε the associated vector field: $(x', y') = F_\varepsilon(x, y)$. Let us note already that when $a < c$ and $b < d$, the point $(1/a, 0)$ attracts any path starting in $(0, +\infty)^2$. We say that the environment is favorable to species x. Similarly, when $a > c$ and $b > d$, the point $(0, 1/d)$ attracts any path starting in $(0, +\infty)^2$. We say that the environment is favorable to species y. See [**?**] for a detailed presentation of the four generic configurations. The environment is said to be of

- Type 1: if $a < c, b < d$ (favorable to species x)
- Type 2: if $a > c, b > d$ (favorable to species y)
- Type 3: if $a > c, b < d$ (persistence)
- Type 4: if $a < c, b > d$ (extinction of species x or y depending on the starting point)

Consider two such systems $\varepsilon_0 = (a_0, b_0, c_0, d_0, \alpha_0, \beta_0)$ and $\varepsilon_1 = (a_1, b_1, c_1, d_1, \alpha_1, \beta_1)$ and introduce the random process $\{(X_t, Y_t, I_t)\}$ on $\mathbb{R} \times \mathbb{R} \times \{0, 1\}$ obtained by switching between these two deterministic dynamics, at rates λ_0, λ_1. More precisely, we consider the Markov process driven by the following generator

$$Lf(z, i) = F_i(z) \cdot \nabla_z f(z, i) + \lambda_i(f(z, 1 - i) - f(z, i)), \quad (z, i) \in \mathbb{R}^2 \times \{0, 1\}.$$

Equivalently, $(I_t)_{t\geq 0}$ is a Markov process on $\{0,1\}$ with jump rate λ_0 and λ_1, that is

$$\mathbb{P}(I_{t+s} = 1-i | I_t = i, \mathcal{F}_t) = \lambda_i s + o(s),$$

where $\mathcal{F}_t$ is the sigma field generated by $\{I_u, u \leq t\}$. Finally, (X_t, Y_t) is solution of

$$(X_t', Y_t') = F_{\varepsilon_{I_t}}(X_t, Y_t).$$

This process on $\mathbb{R}^2 \times \{0,1\}$ has already been studied in [?, ?]. It belongs to the class of the piecewise deterministic Markov processes introduced by Davis [?]. See also [?] for a recent review of the application areas of such processes. Let us introduce the invasion rates of species x and y defined in [?] as

$$\Lambda_y = \int \beta_0(1-c_0x)\mu(dx,0) + \int \beta_1(1-c_1x)\mu(dx,1),$$

$$\Lambda_x = \int \alpha_0(1-b_0y)\widehat{\mu}(dy,0) + \int \alpha_1(1-b_1x)\widehat{\mu}(dy,1),$$

where μ is the invariant probability measure of (X_t, I_t) associated to equation:

$$X_t' = \alpha_{I_t} X_t(1-a_{I_t}X_t),$$

and $\widehat{\mu}$ is the invariant probability measure of (Y_t, I_t) associated to equation:

$$Y_t' = \beta_{I_t} Y_t(1-d_{I_t}Y_t).$$

The meaning of Λ_y is the following: when species y is close to extinction, species x behaves approximately as $(X_t', 0) = F_{\varepsilon_{I_t}}(X_t, 0)$ and Λ_y is the growth rate of species y with respect to invariant measure μ of (X, I). Note that the invasion rates depend on the jump rates $(\lambda_0, \lambda_1) \in (0, +\infty)^2$. For every $(\lambda_0, \lambda_1) \in (0, +\infty)^2$, we have two parametrizations of these jump rates:

$$(s,t) \in [0,1] \times (0,+\infty): \quad st = \lambda_0, \quad (1-s)t = \lambda_1.$$

$$(u,v) \in [0,1] \times (0,+\infty): \quad uv = \lambda_0/\alpha_0, \quad (1-u)v = \lambda_1/\alpha_1.$$

The change of parameters $(u,v) = \xi(s,t)$ is triangular in the sense that u only depends on s

$$(u,v) = \xi(s,t) = \big(\frac{s\alpha_1}{(1-s)\alpha_0 + s\alpha_1}, \frac{t}{\alpha_0\alpha_1}((1-s)\alpha_0 + s\alpha_1)\big).$$

Let us denote the invasion rates in the (u,v) coordinates by

$$\tilde{\Lambda}_x(u,v) = \Lambda_x(\xi^{-1}(u,v)) \quad \text{and} \quad \tilde{\Lambda}_y(u,v) = \Lambda_y(\xi^{-1}(u,v)).$$

It is established in [?] that signs of $\tilde{\Lambda}_x$ and $\tilde{\Lambda}_y$ determine the long time behavior of (X_t, Y_t).

	$\tilde{\Lambda}_y > 0$	$\tilde{\Lambda}_y < 0$
$\tilde{\Lambda}_x > 0$	persistence of the two species	extinction of species y
$\tilde{\Lambda}_x < 0$	extinction of species x	extinction of species x or y

Moreover, in [?] it is shown that two environments of Type 1 may lead to four regimes for the stochastic process. This surprising result is reminiscent of switched stable linear ODE studied in [?, ?].

A fundamental property of the model is that, for all $0 \le s \le 1$, the vector field $(1-s)F_{\varepsilon_0}+sF_{\varepsilon_1}$ is the Lotka-Volterra system associated to the environment $\varepsilon_s = (a_s, b_s, c_s, d_s, \alpha_s, \beta_s)$ with

$$\alpha_s = s\alpha_1 + (1-s)\alpha_0, \quad a_s = \frac{s\alpha_1 a_1 + (1-s)\alpha_0 a_0}{\alpha_s}, \quad b_s = \frac{s\alpha_1 b_1 + (1-s)\alpha_0 b_0}{\alpha_s}, \tag{1.1}$$

$$\beta_s = s\beta_1 + (1-s)\beta_0, \quad c_s = \frac{s\beta_1 c_1 + (1-s)\beta_0 c_0}{\beta_s}, \quad d_s = \frac{s\beta_1 d_1 + (1-s)\beta_0 d_0}{\beta_s}. \tag{1.2}$$

Set

$$I = \{0 \le s \le 1 : a_s > c_s\} \quad \text{and} \quad J = \{0 \le s \le 1 : b_s > d_s\}.$$

We denote by $\tilde{I}$ the image of I for the other parametrization.

Remark 1.1. *As noticed in [**?**], if ε_0 and ε_1 are of Type 1 then I or J may generically be empty or an open interval which closure is contained in $(0,1)$.*

Let us recall below the key result in [**?**] about the expression of the invasion rates.

Lemma 1.2. *[**?**, Lemma 1.2] Assume that ε_0 and ε_1 are of Type 1 and, w.l.g., $a_0 < a_1$. The quantity $\tilde{\Lambda}_y$ can be rewritten as:*

$$\tilde{\Lambda}_y(u,v) = \frac{1}{(a_1-a_0)(\frac{1}{\alpha_0}(1-u)+\frac{1}{\alpha_1}u)}\mathbb{E}[\phi(U_{u,v})]$$

where $\phi : [0,1] \to \mathbb{R}$ is defined by

$$\phi(y) = (a_0 + (a_1-a_0)y)P(\frac{1}{a_0+(a_1-a_0)y}),$$

where

$$P(x) = \Big(\frac{\beta_1}{\alpha_1}(1-c_1x)(1-a_0x) - \frac{\beta_0}{\alpha_0}(1-c_0x)(1-a_1x)\Big)\frac{a_1-a_0}{|a_1-a_0|}, \tag{1.3}$$

and $U_{u,v}$ is a Beta distributed Beta$(uv,(1-u)v)$ random variable. Moreover, ϕ has the following properties:

- *If I is empty then ϕ is nonpositive.*
- *If I is nonempty ($I = (u_1,u_2)$) then ϕ is concave, negative on $(0,u_1)\cup(u_2,1)$ and positive on $\tilde{I} = (u_1,u_2)$.*

Our first result is the precise study of the properties of $\tilde{\Lambda}_x$ and $\tilde{\Lambda}_y$ with two environments $\varepsilon_0, \varepsilon_1$ that are respectively of Type 1 and Type 2. In particular, we describe the regions where $\tilde{\Lambda}_x$ and $\tilde{\Lambda}_y$ are positive.

Theorem 1.3. *(Shape of the regions). Assume that ε_0 and ε_1 are respectively of Type 1 and Type 2. Then, there exists a function $u \mapsto v_y(u)$ from $(0,1) \to [0,\infty]$, such that $\tilde{\Lambda}_y(u,v) < 0$ when $v < v_y(u)$ and $\tilde{\Lambda}_y(u,v) > 0$ when $v > v_y(u)$. Let a be the coefficient of second degree of polynomial P given by (**??**).*

If $a < 0$, there exists $0 < \alpha < \overline{\alpha} < 1$ such that v_y is infinite on $[0,\alpha]$, is decreasing and continuous on $(\alpha,\overline{\alpha})$, tends to $+\infty$ at α, tends to 0 at $\overline{\alpha}$ and is equal to 0 on $[\overline{\alpha},1]$.

If $a > 0$, there exists $0 < \overline{\alpha} < \alpha < 1$ such that v_y is equal to 0 on $[0,\overline{\alpha}]$, is increasing and continuous on $(\overline{\alpha},\alpha)$, tends to 0 at $\overline{\alpha}$, tends to $+\infty$ at α, and is infinite on $[\alpha,1]$.

Moreover, α and $\overline{\alpha}$ are explicit.

The second result is the following theorem.

Theorem 1.4. *For any (i,j) in $\{1,2,3,4\}^2$, there exist two environments ε_0 of Type i and ε_1 of Type j such that the associated stochastic process has four possible regimes depending on the jump rates.*

The paper is organized as follows. In Section 2 we prove the properties of $\tilde{\Lambda}_x$ and $\tilde{\Lambda}_y$. In Section 3 we prove Theorem **??**. In Section 4 we present illustrations obtained by numerical simulation. In Section 5 we study the case when the two environments are of Type 3. Finally, in Section 6, we prove Theorem **??** providing, in each case, a good couple of environments.

2 Expression of invasion rates

Lemma 2.1. *If ε_0 and ε_1 are respectively of Type 1 and Type 2, then $\tilde{I}$ is always nonempty and there exists $0 < \alpha < 1$ (depends on $\alpha_i, \beta_i, a_i, c_i$) such that $\tilde{I} = (\alpha, 1]$.*

Proof. Set

$$R = \frac{\beta_0\alpha_1}{\alpha_0\beta_1}, \quad u = \frac{s\alpha_1}{\alpha_s}, \quad A = (a_1 - a_0)(R-1), \quad B = (2a_0 - c_0 - a_1)R + (c_1 - a_0), \quad C = (c_0 - a_0)R.$$

For any $s \in (0,1)$, we get that

$$c_s - a_s = \frac{Au^2 + Bu + C}{R(1-u) + u}$$

where a_s and c_s are given by (**??**) and (**??**). Set

$$T(u) = Au^2 + Bu + C \quad \forall u \in [0,1].$$

We easily get

$$T(0) = C = (c_0 - a_0)R > 0, \quad T(1) = A + B + C = c_1 - a_1 < 0.$$

Because T is a second degree polynomial with $T(0) > 0$ and $T(1) < 0$, we conclude that

$$T(u) < 0 \Leftrightarrow u > \alpha = \frac{-B - \sqrt{B^2 - 4AC}}{2A}.$$

Therefore $u \in \tilde{I} \Leftrightarrow T(u) < 0 \Leftrightarrow u > \alpha \Leftrightarrow u \in (\alpha, 1]$. As a consequence, $\tilde{I} = (\alpha, 1]$. □

Proposition 2.2. *The map $\tilde{\Lambda}_y(u,v)$ satisfies the following properties:*
For all $u \in [0,1]$

$$\lim_{v\to\infty} \tilde{\Lambda}_y(u,v) = \beta_u(1 - \frac{c_u}{a_u}) \begin{cases} > 0 & \text{if } u \in \tilde{I} = (\alpha, 1], \\ = 0 & \text{if } u \in \partial\tilde{I} = \{\alpha\}, \\ < 0 & \text{if } u \in (0,1) \setminus \overline{\tilde{I}} = [0, \alpha), \end{cases}$$

and

$$\lim_{v\to 0} \tilde{\Lambda}_y(u,v) = \frac{1}{\frac{1}{\alpha_0}(1-u) + \frac{1}{\alpha_1}u} \left(\left(\frac{\beta_1}{\alpha_1}(1 - \frac{c_1}{a_1}) - \frac{\beta_0}{\alpha_0}(1 - \frac{c_0}{a_0}) \right) u + \frac{\beta_0}{\alpha_0}(1 - \frac{c_0}{a_0}) \right). \tag{2.1}$$

Proof. The proposition is obtained by changing variables $(s,t) \longleftrightarrow (u,v)$ from [**?**, Prop. 2.3]. $\square$

Proposition 2.3. *There exists* $0 < \overline{\alpha} < 1$ *such that* $\lim_{v\to 0} \tilde{\Lambda}_y(u,v) > 0$ *if* $u > \overline{\alpha}$ *and* $\lim_{v\to 0} \tilde{\Lambda}_y(u,v) < 0$ *if* $u < \overline{\alpha}$.

Proof. The limit in (**??**) has the same sign than

$$g(u) = \Big(\frac{\beta_1}{\alpha_1}(1-\frac{c_1}{a_1}) - \frac{\beta_0}{\alpha_0}(1-\frac{c_0}{a_0})\Big)u + \frac{\beta_0}{\alpha_0}(1-\frac{c_0}{a_0}), \quad \forall u \in [0,1].$$

We get

$$g(0) = \frac{\beta_0}{\alpha_0}(1-\frac{c_0}{a_0}) < 0 \quad \text{and} \quad g(1) = \frac{\beta_1}{\alpha_1}(1-\frac{c_1}{a_1}) > 0.$$

Since g is a linear function, $\overline{\alpha}$ is the unique zero of g and the result is clear. $\square$

Proposition 2.4. *Let a be the coefficient of degree 2 of polynomial P given by* (**??**)

$$a = \Big(\frac{\beta_1}{\alpha_1}c_1a_0 - \frac{\beta_0}{\alpha_0}c_0a_1\Big)\frac{a_1-a_0}{|a_1-a_0|}.$$

If $a < 0$ (resp. $a > 0$ or $a = 0$) then $\alpha < \overline{\alpha}$ (resp. $\alpha > \overline{\alpha}$ or $\alpha = \overline{\alpha}$).

Proof. By symmetry we only consider the case $a < 0$. Without loss of generality, we assume that $a_1 > a_0$ and a becomes:

$$a = \frac{\beta_1}{\alpha_1}c_1a_0 - \frac{\beta_0}{\alpha_0}c_0a_1.$$

To prove that $\alpha < \overline{\alpha}$, it is sufficient to prove $A\overline{\alpha}^2 + B\overline{\alpha} + C < 0$. Since, by definition of $\overline{\alpha}$,

$$\left(\frac{\beta_1}{\alpha_1}\left(1-\frac{c_1}{a_1}\right) - \frac{\beta_0}{\alpha_0}\left(1-\frac{c_0}{a_0}\right)\right)\overline{\alpha} + \frac{\beta_0}{\alpha_0}\left(1-\frac{c_0}{a_0}\right) = 0,$$

we get, multiplying by $a_0a_1\alpha_1/\beta_1$, that

$$(a_0a_1 - c_1a_0 - Ra_1a_0 + Ra_1c_0)\overline{\alpha} + Ra_1(a_0 - c_0) = 0. \tag{2.2}$$

Replacing $\overline{\alpha}$ by its expression in (**??**), we get:

$$A\overline{\alpha}^2 + B\overline{\alpha} + C = \frac{R(a_1-c_1)(a_0-a_1)(a_0-c_0)(a_0c_1 - Ra_1c_0)}{(a_0a_1 - a_0c_1 - Ra_0a_1 + Ra_1c_0)^2}.$$

Since $c_0 > a_0, a_1 > c_1, a_1 > a_0$ and $a_0c_1 - Ra_1c_0 = \frac{\alpha_1}{\beta_1}a < 0$, we conclude $A\overline{\alpha}^2 + B\overline{\alpha} + C < 0$. $\square$

3 Shape of the positivity region

Recall $a = \Big(\frac{\beta_1}{\alpha_1}a_0c_1 - \frac{\beta_0}{\alpha_0}a_1c_0\Big)\frac{a_1-a_0}{|a_1-a_0|}$ is the coefficient of degree 2 of polynomial P given by (**??**).

Lemma 3.1. *[**?**, Lemma 4.1] Assume ε_0 and ε_1 are of Type 1. If $\tilde{I}$ is nonempty, then the map $(u,v) \to \mathbb{E}[\phi(U_{u,v})]$ is increasing in v and concave in u.*

Remark 3.2. *In Benaïm and Lobry's case, if I is nonempty, ϕ is concave and the parameter a is always negative. In the present case, a may be negative, positive or zero. Therefore, we have the following lemma.*

Lemma 3.3. *Assume ε_0 and ε_1 are respectively of Type 1 and Type 2, then the shape of ϕ depends on the sign of a:*

- *If a is negative, then ϕ is strongly concave and $(u,v) \to \mathbb{E}[\phi(U_{u,v})]$ is increasing in v and concave in u.*
- *If a is positive, then ϕ is strongly convex and $(u,v) \to \mathbb{E}[\phi(U_{u,v})]$ is decreasing in v and convex in u.*
- *If a is zero, then ϕ is linear and $(u,v) \to \mathbb{E}[\phi(U_{u,v})]$ is constant in v and linear in u.*

Proof. This is a straightforward adaptation of [**?**, Lem 4.1]. □

Let us conclude this section with the proof of Theorem **??**.

Proof of Theorem **??**. We consider only the case $a < 0$. Set $K = (\alpha, \overline{\alpha})$. We know clearly that $v \to \tilde{\Lambda}_y(u,v)$ admits:

- negative limits at 0 and ∞ if $u \in [0, \alpha)$,
- positive limits at 0 and ∞ if $u \in (\overline{\alpha}, 1]$,
- a negative limit at 0 and a positive limit at ∞ if $u \in (\alpha, \overline{\alpha})$.

The fact that $v \mapsto \tilde{\Lambda}_y(u,v)$ is increasing justifies the existence of v_y, and we have

$$\tilde{\Lambda}_y(u,v) = 0 \Leftrightarrow u \in K, v = v_y(u).$$

Let us prove that v_y is decreasing in K. Let $\delta_1 < \delta_2$ be two points in K. Choose any $\delta_3 \in (\overline{\alpha}, 1)$, we get $\tilde{\Lambda}_y(\delta_1, v_y(\delta_1)) = 0$ and $\tilde{\Lambda}_y(\delta_3, v_y(\delta_1)) > 0$. Since $\tilde{\Lambda}_x(\cdot, v_y(\delta_1))$ is concave and $\delta_1 < \delta_2 < \delta_3$ we get $\tilde{\Lambda}_y(\delta_2, v_y(\delta_1)) > 0$. Since $\tilde{\Lambda}_y(\delta_2, \cdot)$ is increasing, we obtain $v_y(\delta_2) < v_y(\delta_1)$.

The continuity of v_y on K is a straightforward consequence of the continuity of the function $\tilde{\Lambda}_y$, which is obvious from the expression (**??**).

Let us show v_y tends to ∞ on α. Let $\{u_n\} \subset K : u_n \downarrow \alpha$. Since v_y is decreasing in K, we get $v_y(u_n) \uparrow v \in [0, \infty]$. If v is finite, since the zero set of $\tilde{\Lambda}_y$ is closed, by continuity, $\alpha \in K$ (impossible). So $v_y(u_n) \uparrow \infty$.

Let us prove v_y tends to 0 on $\overline{\alpha}$. Let $\{u_n\} \subset K : u_n \uparrow \overline{\alpha}$. Since v_y is decreasing in K, we get $v_y(u_n) \downarrow \epsilon \in [0, \infty)$. If $\epsilon > 0$, since $u_n < \overline{\alpha}$, we obtain $\tilde{\Lambda}_y(u_n, \epsilon/2) < 0 \ \forall n$. Therefore $0 < \tilde{\Lambda}_y(\overline{\alpha}, \epsilon/2) = \lim_{n \to \infty} \tilde{\Lambda}_y(u_n, \epsilon/2) \leq 0$ (impossible). As a consequence, $\epsilon = 0$ and $v_y(u_n) \downarrow 0$. □

4 Numerical illustrations

Recall that for all $u \in [0,1]$, $v_y(u)$ and $v_x(u)$ are the unique respective solutions of

$$\tilde{\Lambda}_y(u,v) = 0 \quad \text{and} \quad \tilde{\Lambda}_x(u,v) = 0.$$

We now consider, for a varying parameter ρ, the environments

$$\varepsilon_0 = (1, 5, 2, 8, 3, 3) \quad \text{and} \quad \varepsilon_1 = (2, 11, 1, \rho, 2, 1.8). \tag{4.1}$$

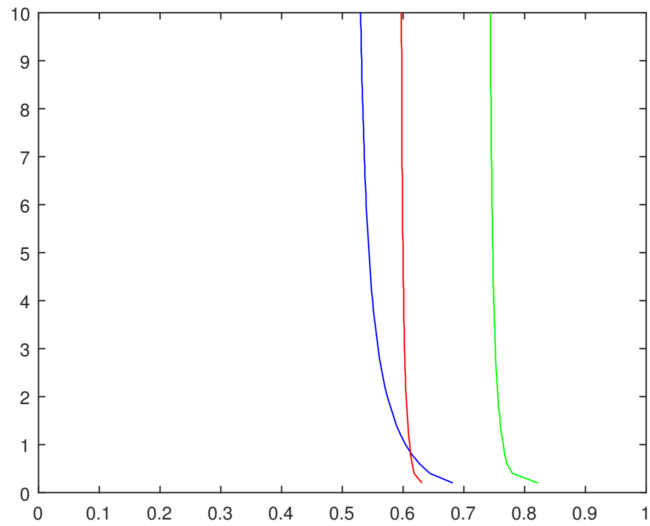

Figure 1: The blue curve is the graph of v_y (it does not depend on ρ); the green and red curves are v_x for the environments given in (**??**) with $\rho = 10$ and $\rho = 9$ respectively.

Figure **??** represents the "critical" functions v_y and v_x for different choices of the environments. Thanks to [**?**], these plots give us information about how many regimes we can observe when the jump rates are modified. For example, the plot for $\rho = 10$ has three regimes: extinction of x (on the right of the green curve), persistence (between the green and blue curves) and extinction of y (on the left of the blue curve). For $\rho = 9$, there is an additional zone (above the red curve and below the blue curve) that corresponds to jump rates leading to random extinction of a species.

5 Switching between two persistent Lotka-Volterra systems

Let us assume that ε_0 and ε_1 are of Type 3. In this case, one can easily get that extinction of species y is not possible if u is to close of 0 or 1; in other words, $[0,1] \setminus \tilde{I}$ is either empty or is an open interval which closure is contained in $[0,1]$. Recall

$$R = \frac{\beta_0 \alpha_1}{\alpha_0 \beta_1}, \quad A = (a_1 - a_0)(R-1), \quad B = (2a_0 - c_0 - a_1)R + (c_1 - a_0), \quad C = (c_0 - a_0)R.$$

Then, we get that

$$[0,1] \setminus \tilde{I} \neq \emptyset \Leftrightarrow \begin{cases} A < 0 \\ \Delta = B^2 - 4AC > 0 \\ 0 < \dfrac{-B - \sqrt{\Delta}}{2A} < 1. \end{cases}$$

Moreover, if $[0,1] \setminus \tilde{I}$ is nonempty, then the map $(u,v) \to \mathbb{E}[\phi(U_{u,v})]$ is (strictly) decreasing in v and convex in u. This is a straightforward adaptation of Lemma 4.1 in [**?**].

Figure **??** provides the shape of v_x and v_y for the environments $\varepsilon_0 = (6,1,4,2,1,5)$ and $\varepsilon_1 = (3,3,2,5.5,5,1)$. Once again, the switched process has four regimes depending on the jump rates.

Remark 5.1. *We see a surprising result : although both vector fields are persistent, the stochastic process may lead to the extinction of one of the two species.*

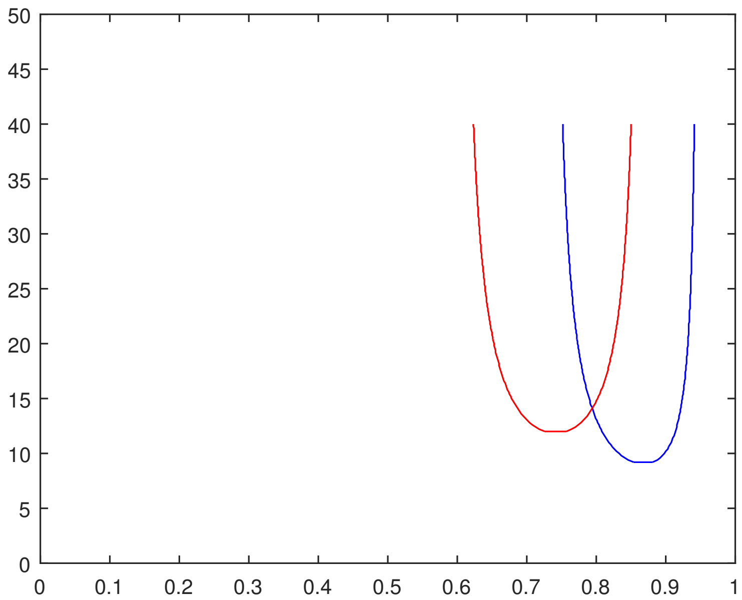

Figure 2: Graph of v_y (blue curve) and v_x (red curve) for the environments $\varepsilon_0 = (6, 1, 4, 2, 1, 5)$ and $\varepsilon_1 = (3, 3, 2, 5.5, 5, 1)$.

6 General case: proof of Theorem ??

The following array presents, for any couple of types, an example of two environments that are associated to a stochastic process with four regimes depending on the jump rates. The first line has been obtained in [**?**]. The second line is studied in Section 2. The fifth line is studied in Section 5. The reader can easily check that the other cases correspond to Figure **??**.

(F_0, F_1)	a_0	b_0	c_0	d_0	α_0	β_0	a_1	b_1	c_1	d_1	α_1	β_1
Type 1-1	1	1	2	2	1	5	3	3	4	3.5	5	1
Type 1-2	1	5	2	8	3	3	2	11	1	9	2	1.8
Type 1-3	1	1	3.5	2	1	5	5	3	4	5.5	5	1
Type 1-4	1	1	2	3.5	1	5	3	4	4	3	5	1
Type 3-3	6	1	4	2	1	5	3	3	2	5.5	5	1
Type 3-4	6	1	4	8	1	5	3	10	4	7	5	1
Type 4-4	2	2	1	1	5	1	7	3.5	4	3	1	5

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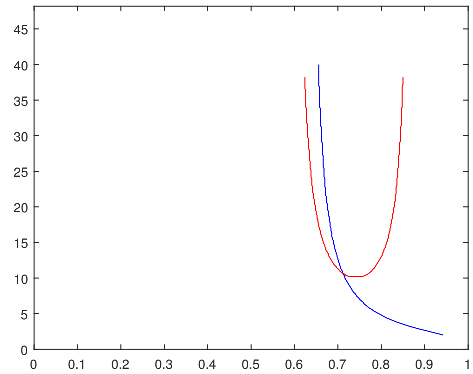
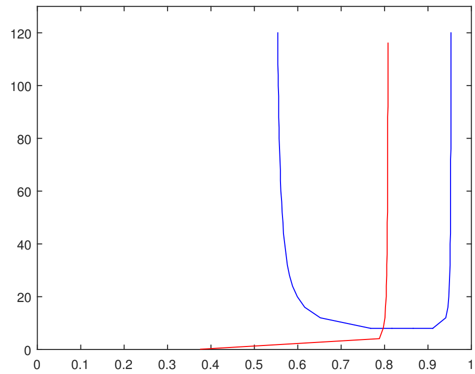
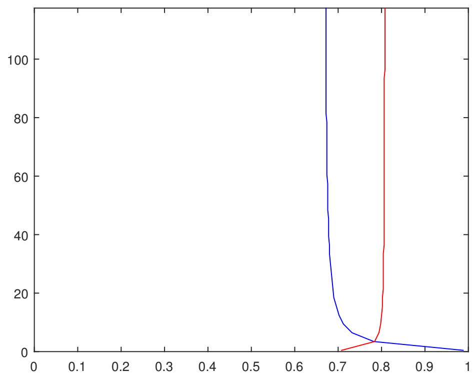


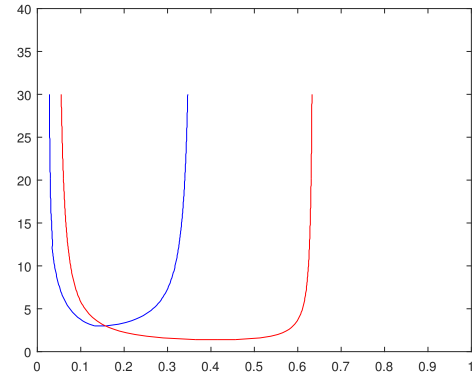
(a) Type 1-3



(b) Type 1-4



(c) Type 3-4



(d) Type 4-4

Figure 3: Graph of v_y (blue curve) and v_x (red curve) for the four last cases.

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Florent Malrieu, e-mail: `florent.malrieu(AT)univ-tours.fr`

Laboratoire de Mathématiques et Physique Théorique (UMR CNRS 7350), Fédération Denis Poisson (FR CNRS 2964), Université François-Rabelais, Parc de Grandmont, 37200 Tours, France.

Tran Hoa Phu, e-mail: `thphu1(AT)yahoo.com`

Laboratoire de Mathématiques et Physique Théorique (UMR CNRS 7350), Fédération Denis Poisson (FR CNRS 2964), Université François-Rabelais, Parc de Grandmont, 37200 Tours, France.